The impact of the underlying interest rate process

- When calculating the best estimate of liabilities

By Johan Dellner

Insurance companies use different types of interest rate models for the same type of valuation, the valuation of the time value of options and guarantees (TVOG). This paper aims to explain and examine the impact of using the following underlying interest rate models: Hull and White (HW); Cox–Ingersoll–Ross (CIR); and Libor Market Model (LMM) when generating the economic scenarios used for the valuation. These three processes are all used in the insurance industry and fulfill to a certain extent the market consistency and risk neutrality required by EIOPA under Solvency II.

The differences in their distributions yield different results when the TVOG is determined, which indicates the importance of using the appropriate model as well as understanding it. The TVOG is significantly different for products with a guaranteed rate of 0% due to the allowance of negative rates in HW.

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1 Introduction

The upcoming European regulation Solvency II increases the overall effort insurance companies' needs to put into their regulatory reporting. This paper will focus on one of the new requirements: the calculation of the time value of options and guarantees (TVOG) and their dependence on the choice of asset model.

The purpose of the TVOG is to reflect the value of the uncertainty of meeting the obligations the insurance company has taken by promising an investment allocation together with future guaranteed amounts to their policyholders – guarantees which create an option value due to their asymmetry. A typical example is an insurance contract with profit participation and a maturity guarantee. If the insurance company earns an investment return which is in excess of the guarantee, the policyholder will get a discretionary amount in addition. However, if the investment return is insufficient to cover the guaranteed amount, the insurance company has to cover the loss. This creates the asymmetry which creates the option value. Hull (2006) describes option values in great detail in his book *Options, Futures, and Other Derivatives*.

Many articles, such as Li and Zhao (2006), have been written discussing the difficulties of replicating market prices and complex models have been introduced to capture the behavior of the market. There exist very few articles (none having been found during the duration of this work) which examine the prospective impact of the underlying interest rate models for life insurance contracts, i.e. the valuation of the TVOG. This paper addresses this area and gives an answer to the question of whether the choice of interest rate model impacts the valuation of the best estimate of liability.

Section 2 describes the three interest rate models used to illustrate this problem. Section 3 shows the output for 3000 scenarios for each model and then Section 4 shows the TVOGs for a sample guaranteed product using those stochastic scenarios. Section 5 draws conclusions.

2 Interest rate models

There are several different providers of economic scenario generators (ESG) in Europe as described by InsuranceERM (2013). These ESGs are in many cases based on different underlying interest rate models. The two main requirements for scenarios used to determine the TVOG is that they are market consistent and risk neutral. In short, market consistent refers to the reproduction of any price observed in the market and risk neutral refers to being arbitrage free by fulfilling the martingale test (1=1). The following underlying interest rate processes: Hull and White (HW), Cox–Ingersoll–Ross (CIR) and Libor Market Model (LMM) all fulfill these demands to a certain extent. All are used in the insurance industry.

It is clear from the structure of the interest rate processes in these models that the distribution of resulting interest rates will be very different. While HW allows for negative interest rates, the CIR and LMM do not. The CIR uses a fixed volatility term while HW and LMM allows for fluctuation of the volatility term structure. This implies that it is significantly more difficult to replicate the market's option price surface using CIR, compared to HW and LMM.

2.1 Cox-Ingersoll-Ross model

Cox-Ingersoll-Ross (CIR) is a one factor interest rate model and was introduced by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross in 1985. Its short rate dynamics is given by the stochastic differential equation:

$$dr(t) = a\big(\beta - r(t)\big)dt + \sigma\sqrt{r(t)}dW(t)$$

Where:

$r(t)$	= Short rate
a	= Short rate reversion parameter
β	= Long term mean
σ	= Volatility
$W(t)$	= Wiener process

The square root of the short rate guarantees the short rate will never turn negative.

2.2 Hull and White model

There are several different types of Hull and White models (HW); the description here will be limited to the one-factor HW (also known as the extended Vasicek model) which was introduced by John Hull and Alan White in 1993. The short rate dynamics are given by the stochastic differential equation:

$$dr(t) = a\big(\vartheta(t) - r(t)\big)dt + \sigma(t)dW(t)$$

Where:

$r(t)$	= Short rate
a	= Short rate reversion parameter
$\vartheta(t)$	= Long term mean
$\sigma(t)$	= Volatility
$W(t)$	= Wiener process

The short rate reversion parameter adjusts the speed of reversion towards the long term mean; the long term mean is set by the initial yield curve. A piecewise volatility is calibrated in the economic scenario generator (ESG) with the following structure:

$$\sigma(t) = \begin{cases} \sigma_1 & if\ 0 \le t < 1 \\ \sigma_2 & if\ 1 \le t < 2 \\ \sigma_3 & if\ 2 \le t < 3 \\ \sigma_4 & if\ 3 \le t < 4 \\ \sigma_5 & if\ 4 \le t < 5 \\ \sigma_7 & if\ 5 \le t < 7 \\ \sigma_{10} & if\ 7 \le t < 10 \\ \sigma_{15} & if\ 10 \le t < 15 \\ \sigma_{20} & if\ 15 \le t < 20 \\ \sigma_{25} & if\ 20 \le t < 25 \\ \sigma_{30} & if\ 25 \le t < 30 \end{cases}$$

2.3 Libor Market Model

Brigo and Mercurio (2001) explored the Libor Market Model (LMM). It is fundamentally different to the models of HW and CIR. LMM captures the dynamics of the entire yield curve by using building blocks of forward rates. They define the standard lognormal LMM model as:

$$dF_i(t) = \mu_i(F(t), t)dt + F_i(t)\sigma_i(t)dW(t)$$

Where:

$F_i(t)$ = Forward rates

$\mu(F, t)$ = The drift of forward rate Fi

$\sigma_i(t)$ = Volatility

$W(t)$ = Wiener process

In practice, the standard lognormal LMM fails to capture the volatility smile in derivatives markets. However, by adjusting the volatility within the LMM to the SABR volatility model, it can be accomplished. Hagan and Lesniewski (2008) derive the SABR-LMM by first defining the SABR volatility model using the following system of stochastic differential equations:

$$dF_i(t) = F_i(t)^{\beta}\sigma_i(t)dW(t)$$

$$d\sigma_i(t) = \alpha\sigma_i(t)dZ(t)$$

Where:

$Z(t)$ = Wiener process with correlation coefficient -1 < p < 1 with $W(t)$.

α = A constant parameter, > 0

β = A constant parameter, 1 > b > 0

This yields the SABR-LMM which is the one used in the scenario generation:

$$dF_i(t) = \mu_i^{SABR}(F(t), \sigma(t), t)dt + F_i(t)^{\beta}\sigma_i(t)\sum_{j=1}^{N_f} b_{ij}\, dZ_j(t)$$

$$d\sigma_i(t) = \tau_i^{SABR}(F(t), \sigma(t), t)dt + \varphi_i(t)\sum_{j=1}^{N_f} c_{ij}\, dW_j(t)$$

Where:

μ_i^{SABR} = The drift of forward rate Fi in SABR-LMM

τ_i^{SABR} = The drift of the volatility in SABR-LMM

b_{ij} = A time independent parameter

c_{ij} = A time independent parameter

$\varphi_i(t)$ = A time dependent exponential function

3 Analysis of generated scenarios

The three models were parameterized using market data from Bloomberg as of 2013-06-30 in order to generate three sets of 3 000 scenarios each, i.e. one set for each model. All market data were in SEK.

3.1 Risk neutrality and market consistency

In general, scenario sets must satisfy a number of conditions to be used for TVOG valuation. For example:

- All investment strategies must, “on average”, be equivalent to a risk-free investment
- The yield curve derived from the average discount factor must be equal to the market yield curve
- The scenarios must replicate swaption prices
- The scenarios must replicate index asset-option prices
- The scenarios must reflect the markets’ expectations (e.g. inflation)

In addition to these conditions, other validations and checks were performed to confirm that the scenario sets reflect the market (e.g. graphical inspection of distributions).

As only the interest rate models are included in this work and, more precisely, only the short rate, these checks were limited to the implied yield curve, swaption prices and distributions.

3.2 Is the yield curve replicated?

The implied yield curve from the different scenario sets is calculated by averaging the discount factors in the scenario sets. Once the discount factors have been determined the spot rates can be calculated.

$$\beta(t) = E[\beta(t)_i], \ r(t) = \beta(t)^{-\frac{1}{t}} - 1$$

Where;

$\beta(t)_i$ = Discount factor at time t and interation i in a scenario set.

$r(t)$ = The implied spot rate at time t.

It is difficult to see any differences by visual inspection; they are all very close to one another.

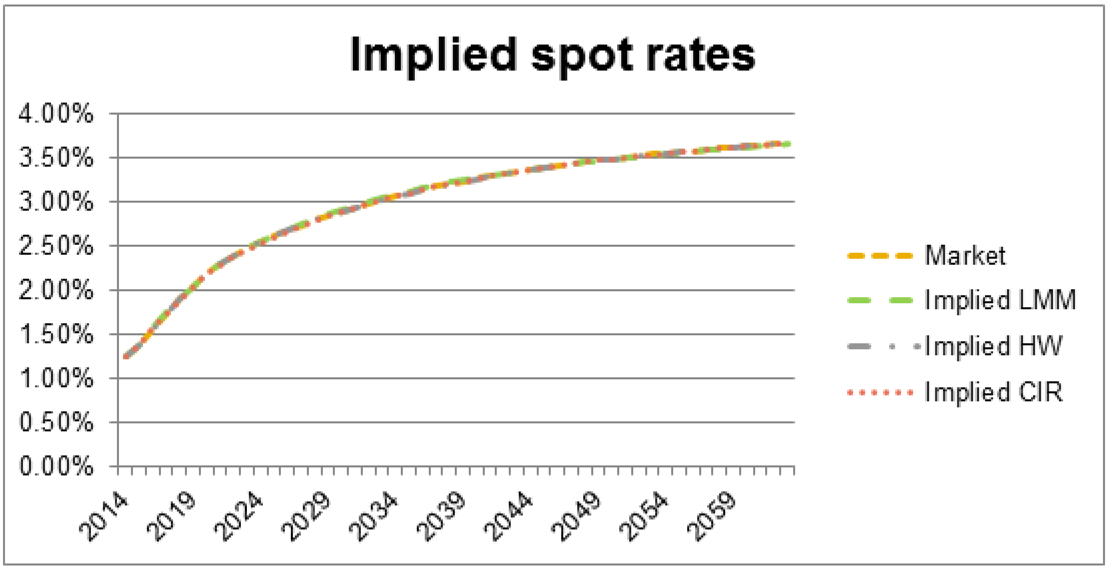

Figure 1: The implied yield curve using the different interest models together with the market yield curve; which was used during the calibration processes.

3.3 Are the swaption prices replicated?

Swaption prices were calculated based on the scenario set and compared to market prices. As illustrated in Figures 2, 3 and 4, the differences are significant and, as expected, the CIR-process fails to capture the full volatility surface. This is mostly due to the fact that CIR uses a single volatility factor.

Total error (%) - Market price vs Scenario price

Option Expiry (Years)	Swap Tenor (Years)												
	1	2	3	4	5	6	7	8	9	10	15	20	25	30
1	-10.62%	-30.79%	-39.47%	-41.39%	-41.80%	-40.23%	-37.94%	-35.55%	-31.98%	-28.41%	-7.20%	7.53%	17.40%	24.50%
2	-34.94%	-42.13%	-43.33%	-42.40%	-40.65%	-37.36%	-33.57%	-29.14%	-23.95%	-18.05%	10.27%	28.07%	37.97%	43.46%
3	-46.72%	-46.04%	-44.44%	-41.46%	-37.61%	-32.91%	-27.56%	-21.14%	-15.08%	-8.57%	23.00%	40.48%	47.93%	50.64%
4	-49.89%	-48.45%	-45.17%	-40.14%	-34.08%	-27.58%	-20.03%	-12.72%	-5.70%	1.11%	33.50%	48.42%	52.15%	53.17%
5	-50.05%	-47.82%	-43.19%	-37.03%	-29.23%	-21.07%	-12.43%	-4.41%	3.30%	10.49%	42.74%	55.48%	56.99%	55.32%
7	-48.01%	-46.03%	-40.13%	-30.91%	-19.37%	-9.79%	-0.75%	7.52%	14.81%	21.50%	48.67%	60.50%	58.48%	53.43%
10	-47.41%	-42.70%	-34.62%	-24.12%	-11.41%	1.16%	12.81%	23.06%	31.96%	39.56%	68.40%	70.91%	67.47%	60.85%
15	-51.61%	-45.62%	-35.08%	-22.28%	-8.51%	4.03%	16.12%	26.49%	35.06%	42.33%	65.96%	67.27%	63.05%	56.80%
20	-56.79%	-52.06%	-38.22%	-21.85%	-6.45%	6.65%	17.15%	28.16%	36.93%	44.55%	64.58%	71.23%	65.05%	58.02%
25	-61.80%	-53.10%	-36.26%	-19.16%	-2.92%	10.96%	20.36%	31.64%	40.70%	48.68%	74.30%	76.76%	68.29%	60.33%
30	-62.07%	-49.48%	-27.57%	-6.86%	12.05%	28.71%	39.12%	51.60%	61.37%	69.14%	90.90%	87.25%	76.00%	61.85%

Figure 2: Total error (%) between Market price and Scenario price for CIR.

Total error (%) - Market price vs Scenario price

Option Expiry (Years)	Swap Tenor (Years)												
	1	2	3	4	5	6	7	8	9	10	15	20	25	30
1	39.64%	8.15%	-6.03%	-10.32%	-12.68%	-12.55%	-11.85%	-11.45%	-9.85%	-8.60%	-1.70%	-2.16%	-4.20%	-5.74%
2	17.00%	3.07%	-1.77%	-4.13%	-6.08%	-6.34%	-6.51%	-6.25%	-5.37%	-3.93%	1.16%	0.08%	-2.05%	-3.56%
3	-0.40%	-0.50%	-1.15%	-1.60%	-2.00%	-2.30%	-2.58%	-1.99%	-2.03%	-1.61%	1.81%	0.53%	-1.84%	-3.43%
4	-2.99%	-2.43%	-1.74%	-0.73%	-0.02%	-0.09%	0.48%	0.31%	-0.19%	-0.64%	1.46%	-0.75%	-3.97%	-5.08%
5	-1.70%	-0.83%	0.42%	1.21%	2.28%	2.39%	2.43%	1.69%	0.98%	0.32%	1.88%	-0.42%	-3.31%	-4.49%
7	-2.89%	-4.10%	-3.10%	-0.82%	1.42%	-0.02%	-1.83%	-3.55%	-5.19%	-6.39%	-7.03%	-6.86%	-8.75%	-9.52%
10	0.39%	2.51%	2.69%	1.67%	1.18%	0.34%	-0.43%	-1.13%	-1.77%	-2.36%	-0.40%	-2.16%	-1.42%	-0.43%
15	6.91%	6.83%	4.95%	1.89%	-0.19%	-1.43%	-1.39%	-1.41%	-1.59%	-1.60%	2.14%	4.48%	8.35%	11.58%
20	-1.29%	-7.04%	-6.98%	-7.66%	-7.59%	-7.88%	-8.63%	-7.53%	-6.87%	-5.89%	-0.32%	8.99%	14.01%	17.69%
25	-10.23%	-10.90%	-11.02%	-11.52%	-10.71%	-10.24%	-11.52%	-9.84%	-8.47%	-6.71%	5.56%	15.41%	20.46%	23.91%
30	-11.16%	-9.82%	-8.43%	-7.40%	-5.97%	-4.48%	-5.59%	-3.50%	-1.73%	0.08%	12.95%	22.44%	28.04%	28.16%

Figure 3: Total error (%) between Market price and Scenario price for HW.

Total error (%) - Market price vs Scenario price

Option Expiry (Years)	Swap Tenor (Years)												
	1	2	3	4	5	6	7	8	9	10	15	20	25	30
1	8.72%	0.27%	-4.11%	-4.62%	-5.38%	-4.16%	-2.87%	-2.42%	-1.12%	-0.39%	3.97%	1.54%	-2.16%	-5.54%
2	4.18%	0.57%	-0.71%	-1.96%	-3.57%	-3.83%	-4.44%	-5.05%	-5.08%	-4.49%	-2.56%	-5.41%	-9.05%	-12.23%
3	-3.87%	-0.56%	-0.04%	-0.07%	-0.74%	-1.67%	-3.06%	-3.63%	-4.74%	-5.31%	-5.24%	-8.24%	-11.99%	-15.16%
4	-4.35%	-1.86%	-0.10%	1.02%	1.02%	-0.37%	-1.16%	-2.50%	-4.09%	-5.55%	-7.04%	-10.73%	-14.84%	-17.49%
5	-1.34%	1.54%	2.99%	2.94%	2.51%	1.13%	-0.15%	-2.08%	-3.86%	-5.46%	-6.96%	-10.46%	-14.34%	-17.27%
7	6.22%	4.51%	4.14%	4.84%	5.56%	2.56%	-0.51%	-3.25%	-5.68%	-7.48%	-9.91%	-11.29%	-14.65%	-17.73%
10	5.30%	7.24%	6.64%	4.73%	3.37%	1.71%	0.37%	-0.76%	-1.68%	-2.49%	-1.86%	-5.47%	-7.73%	-9.72%
15	7.89%	7.49%	4.88%	1.34%	-1.08%	-2.63%	-3.02%	-3.46%	-4.14%	-4.71%	-3.94%	-5.90%	-6.33%	-6.36%
20	5.38%	-1.59%	-2.39%	-4.00%	-4.70%	-5.75%	-7.30%	-6.92%	-6.92%	-6.69%	-6.80%	-3.49%	-2.38%	-1.73%
25	-1.51%	-3.16%	-4.01%	-5.41%	-5.52%	-6.30%	-9.11%	-8.96%	-9.06%	-8.71%	-2.99%	2.28%	3.92%	4.53%
30	-0.15%	-1.39%	-2.13%	-3.04%	-3.52%	-3.89%	-6.62%	-5.89%	-5.42%	-4.69%	3.08%	8.11%	9.59%	7.06%

Figure 4: Total error (%) between Market price and Scenario price for LMM.

3.4 The distributions

By using the scenario sets to build up short rate indices, i.e. setting an index to 1 at the calibration date and then credit it with the short rate, a histogram of those indices after 30 years can be used to illustrate the nature of the models as shown in Figures 5 to 7. The non-negative rates for CIR and LMM prevent them from ending up with an index below one. Note that the mean of the inverse of these indices are equal to each other for the different interest rate models, as implied by Figure 1.

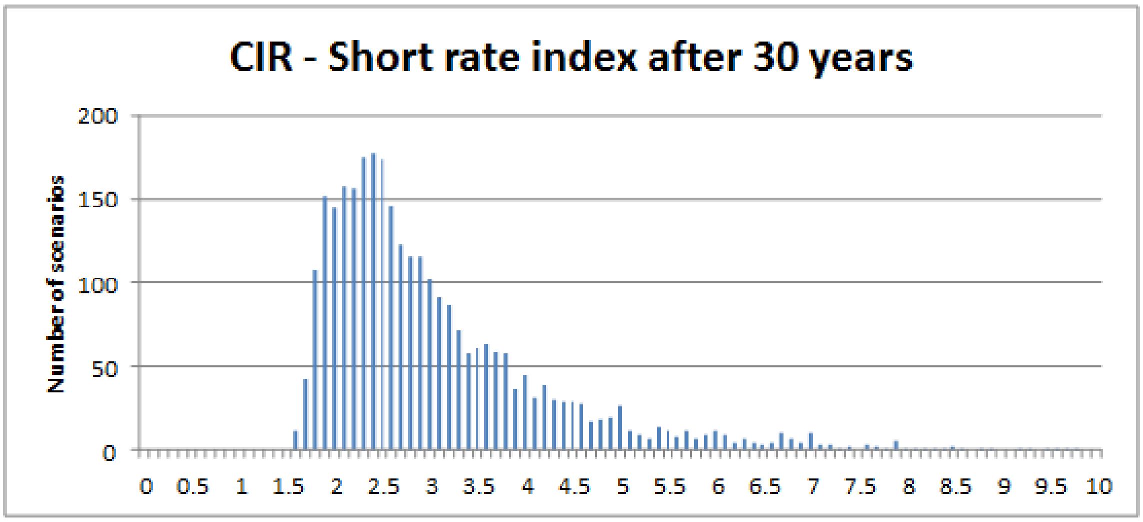

Figure 5: A histogram of the CIR short rate index after 30 years with 3 000 scenarios.

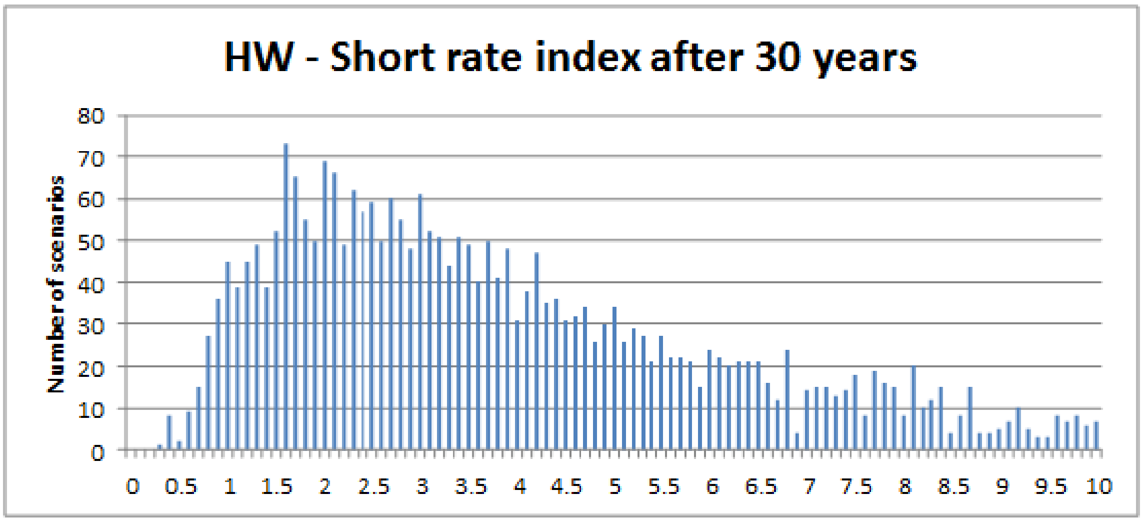

Figure 6: A histogram of the HW short rate index after 30 years with 3 000 scenarios.

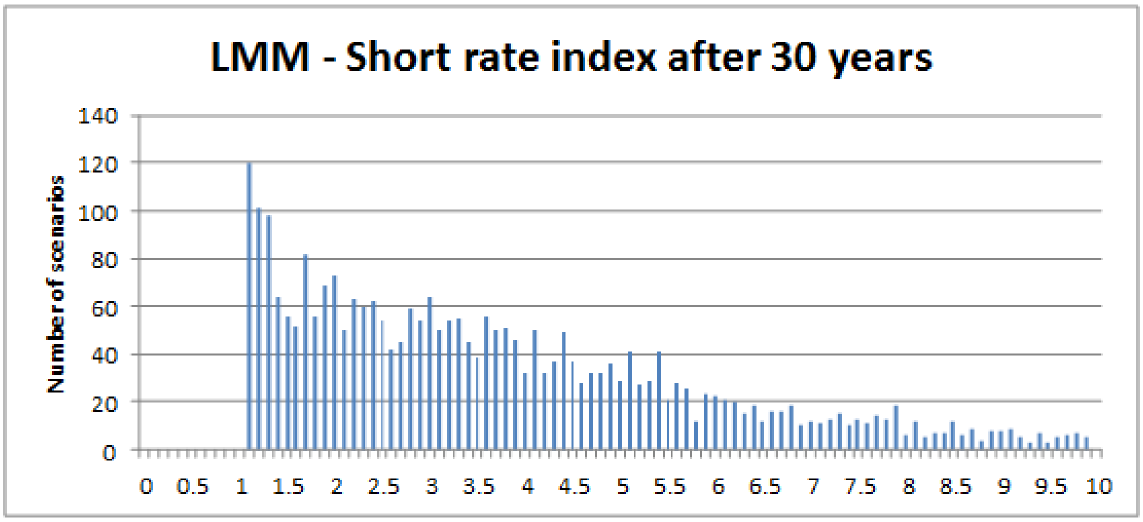

Figure 7: A histogram of the LMM short rate index after 30 years with 3 000 scenarios.

4 TVOGs for a sample life insurance product

The generated scenarios have been applied in the calculation of the TVOG for a simple guaranteed product. Three different guarantee rates have been used 0%, 2% and 4% pa.

4.1 The guarantee product

The policyholder will receive a lump sum payment at a fixed maturity date; the lump sum payment will be the maximum of the fund value and the guaranteed amount at maturity. This is illustrated below. The guarantee holds if he dies before maturity, but if he surrenders he will only receive the fund value. Maturity is assumed to be at age 65. The annual premium assumed is 1000 SEK.

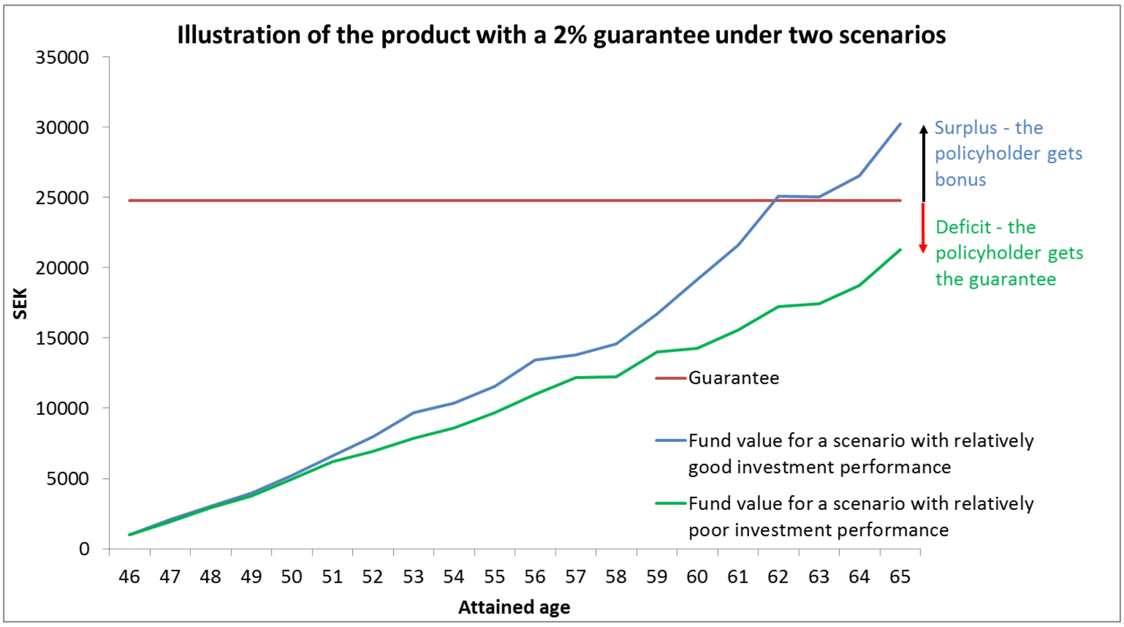

Figure 8: An illustration of the product with a 2% guarantee under two scenarios.

All administration charges are ignored. There is an annual guarantee charge of 0.2% on the fund value.

The Swedish mortality table M90 for male lives, a flat paid-up rate of 2% and flat surrender rate of 1% are applied. The entire fund value is assumed to be invested in short duration bonds and credited interest based on the short rate.

4.2 Results

Figure 9 illustrates the TVOG for individuals entering a policy at different ages. With a guaranteed rate of 0%, only HW ends up with a significant TVOG due to the fact that it is the only model of the three that allows for negative interest rates. The guarantee charge is the only reason LMM and CIR end up with an option value.

The downward slope of the curve is caused by two main factors: policyholders entering the contract at a high age have paid fewer premiums by the time of retirement and the spread of the different outcomes in the scenario set grows larger over time.

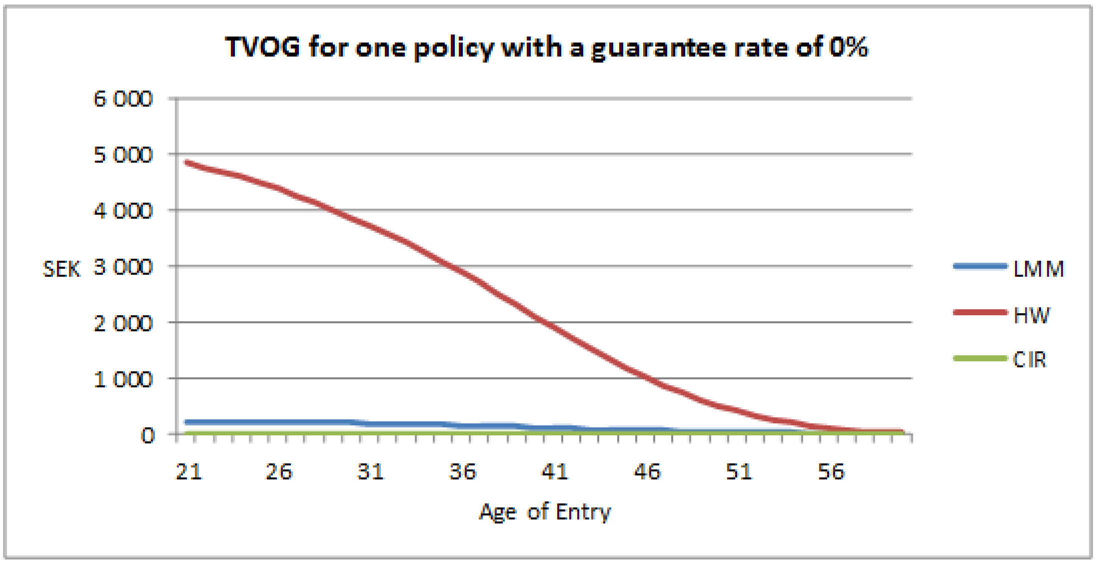

Figure 9: An illustration of how the TVOG varies between different ages and models with a guaranteed rate of 0%.

Figure 10 shows the result for a 2% annual guarantee and follows a similar slope to Figure 9. The TVOG calculated using the LMM model generates a lower value than HW while the CIR model generates a value close to 0.

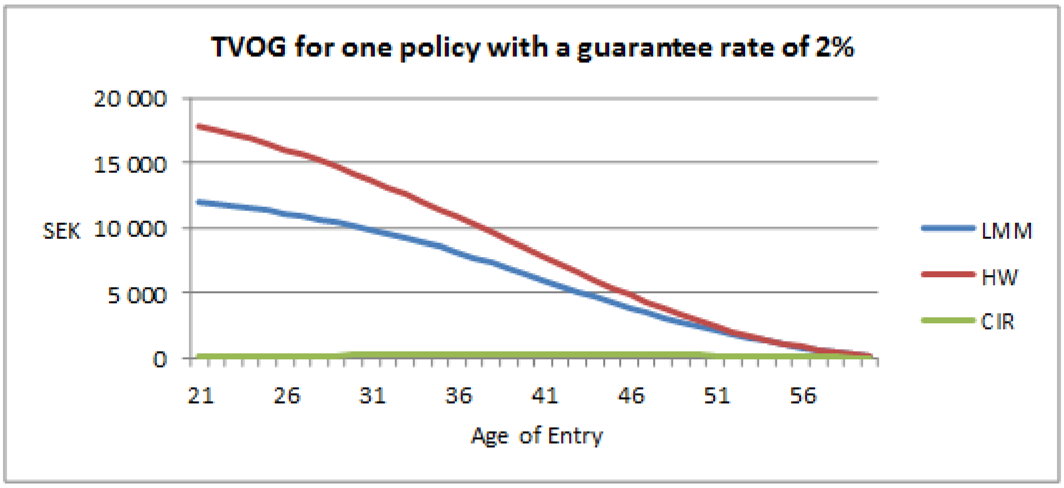

Figure 10: An illustration of how the TVOG varies between different ages and models with a guaranteed rate of 2%.

With a guaranteed rate of 4%, figure 11 shows that the HW model still generates the largest TVOG. The values are closer to LMM for the policyholders entering the policy at a higher age. For CIR the values have increased compared to the previous guaranteed rates of 0% and 2%, but they are still significantly lower than both HW and LMM.

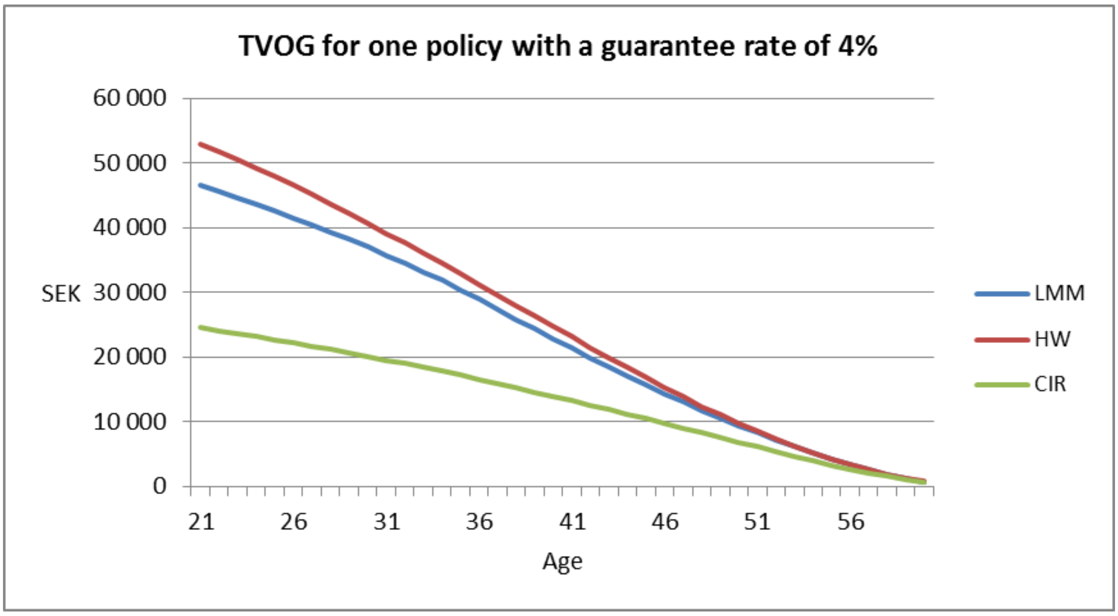

Figure 11: An illustration of how the TVOG varies between different ages and models with a guaranteed rate of 4%.

By applying flat rates of 0%, 2% and 4% to produce short rate indices as illustrated in Figure 12, the outcome of the different TVOGs becomes even clearer. Only the scenarios to the left of the red lines give rise to a TVOG.

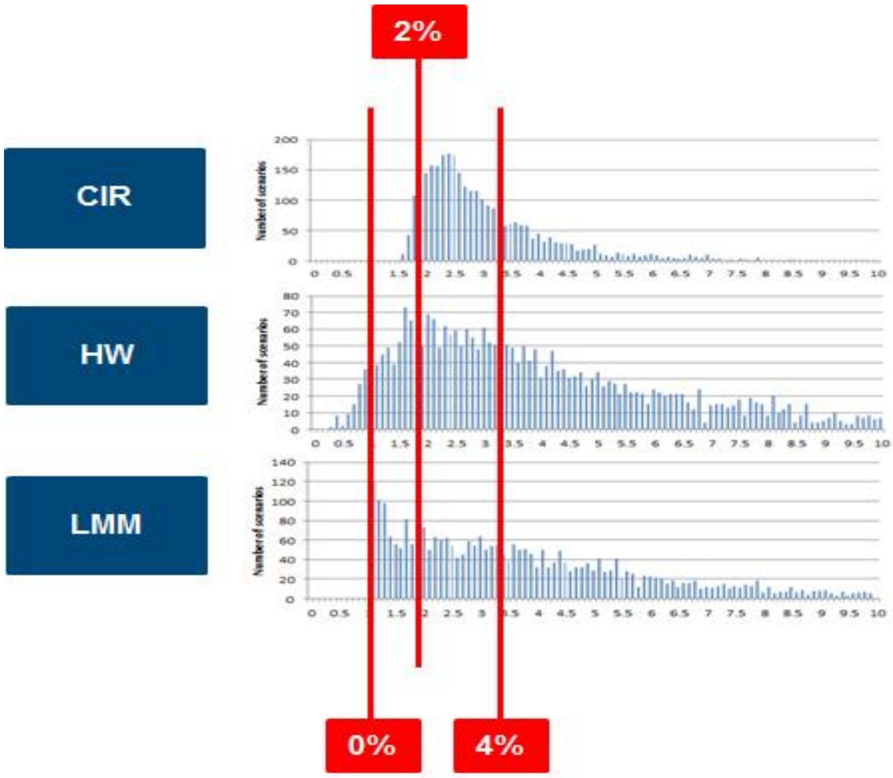

Figure 12: The distributions of the short rate index after 30 years with the different models with red markers to indicate the level of the index with flat rates of 0%, 2% and 4%.

5 Discussion and conclusion

From the above investigation, we conclude that:

- The choice of interest rate model used for the valuation impacts the TVOG significantly.

- The differences between the interest rate models seem to be the most significant for products with a guaranteed rate of 0% due to the allowance of negative rates in HW.

- The interest rate models fail to fully replicate market prices, especially the CIR.

It is important to understand the differences and be aware of the models' limitations when applying them – an interest rate model used properly for one purpose does not necessarily need to be good for another. This work is limited to a very particular guaranteed product, three different interest rate models with certain calibration modifications and market data from a specific date. The differences in TVOG illustrated can be a result of any of these factors – in our example HW always ends up with a higher option value than both LMM and CIR. The conclusion should not be that HW will always generate a higher TVOG; it is rather the fact that these different models will result in different option values and this will directly have an impact on the Solvency II balance sheet.

In general, equities have a higher volatility than bonds and as the volatility is the main source of any financial option value one of the limitations in this work is to assume 100% short duration bonds when investigating the sensitivities of the TVOG. Nevertheless, it is still the interest rate models that underly most index asset models that are created to model equity returns and the cash flows will always be discounted by the short rate.

It is noted that the CIR model uses a single volatility parameter and therefore fails to capture the full volatility surface of the market. As a result, it is very difficult to draw any conclusion in regards to the differences in the generated TVOG using CIR compared to HW and LMM other than the fact that using CIR with a single volatility parameter is too simplistic. CIR could potentially be used to capture the TVOG by using different sets of stochastic scenarios depending on the duration of guarantees.

Finally, the focus in this work has been to present and illustrate that the choice of interest model is important and will affect the results. It is left for the reader to select which model would suit his purpose.

6 References

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